

# Chiral Random Matrix Models: A Novel Intermediate Asymptotic Regime

N. Deo<sup>1,2</sup>

<sup>1</sup> Poornaprajna Institute of Scientific Research  
4 Sadashiva Nagar, Bangalore 560080, India

<sup>2</sup> Abdus Salam International Center  
for Theoretical Physics, Trieste, Italy  
ndeo@vsnl.net

## Abstract

The Chiral Random Matrix Model or the Gaussian Penner Model (generalized Laguerre ensemble) is re-examined in the light of the results which have been found in double well matrix models [1, 2] and subtleties discovered in the single well matrix models [3]. The orthogonal polynomial method is used to extend the universality to include non-polynomial potentials. The new asymptotic ansatz is derived (different from Szego's result) using saddle point techniques. The density-density correlators are the same as that found for the double well models ref. [2] (there the results have been derived for arbitrary potentials). In the smoothed large  $N$  limit they are sensitive to odd and even  $N$  where  $N$  is the size of the matrix [2]. This is a more realistic random matrix model of mesoscopic systems with density of eigenvalues with gaps. The eigenvalues see a brick-wall potential at the origin. This would correspond to sharp edges in a real mesoscopic system or a reflecting boundary. Hence the results for the two-point density-density correlation function may be useful in finding one eigenvalue effects in experiments in mesoscopic

systems or small metallic grains. These results may also be relevant for studies of structural glasses as described in ref. [4].

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# 1 Introduction

In this work we re-visit the Penner model in the context of disordered mesoscopic systems where they naturally appear e.g. in disordered models of metal-insulator transitions [6, 7], superconducting-normal interface with Andreev reflection (the opening of a gap) [8] and also as models of structural glasses [4]. Historically the Penner matrix model was studied in the context of the moduli space of a punctured surface [9]. There an equality between the Penner matrix model and the Euler characteristic of moduli space of punctured surfaces was computed, before taking the continuum limit. Later in [10] it was shown that even after taking the continuum limit the Euler characteristic of moduli space of unpunctured surfaces was obtained as the free energy of the Penner model. This was done in an effort to understand 2-dimensional gravity coupled to matter at the critical point  $c=1$ . Recent advances in this direction are contained in [11]. There has also been recent work on the generalized Laguerre ensemble in the context of the Chiral Random Matrix models of QCD (see [12, 13] and references therein) and in describing a novel group structure associated with scattering in disordered mesoscopic wires [7].

This study focuses on the Gaussian Penner model with potential  $V(M) = \frac{1}{2}\mu M^2 - \frac{t}{2}\ln M^2$  where  $M$  is a  $N \times N$  random matrix; after some work on gapped matrix models has been reported and clarified [1, 2, 4]. In the recent work two basic ideas have been implemented. First the idea of symmetry breaking (see [5, 4]) and then a new asymptotic ansatz of the orthogonal polynomials [1]. Here for the Gaussian Penner model the corresponding polynomials are the associated Laguerre polynomials,  $L_N^\alpha(x)$ , where both the  $\alpha$  and  $N$  asymptotes are to be taken, in previous work only the  $N$  asymptote was found. It turns out that it is in this asymptotic region that the singular models make contact with the double well random matrix models and the universality of the gapped matrix model is extended to include non-polynomial potentials. Ideas of symmetry breaking are also true for the Gaussian Penner model [5, 4] but because of the brick wall potential at the origin, which involves one electron quantum tunneling through an infinite barrier, it is harder to give an intuitive picture. These ideas will be elaborated on in future papers.

The paper is divided as follows. It starts by describing completely the model and establishing the notation and conventions. In section 2, the old asymptotic ansatz of Szego for the associative Laguerre polynomials is taken

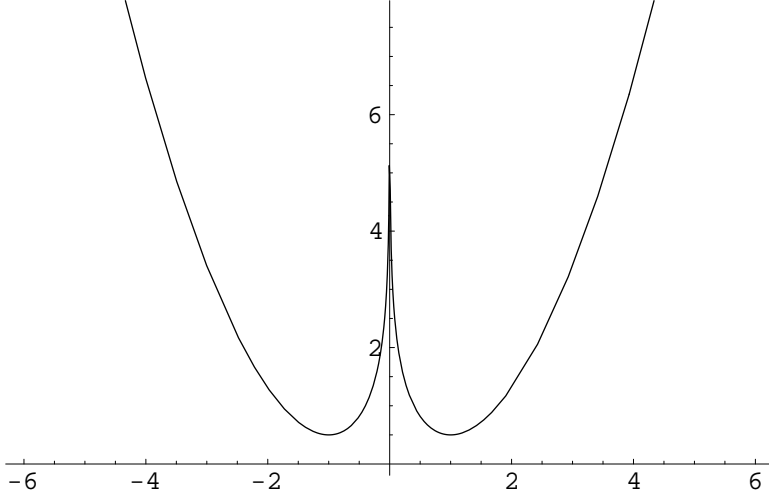


Figure 1: The potential for the Gaussian Penner Random Matrix Model with  $V(M) = \frac{1}{2}\mu M^2 - \frac{t}{2} \ln M^2$ ,  $\mu = 1$  and  $t = 1$ .

and shown to not correspond to the new universality of the double well matrix model studied in [1, 2]. In section 3, the new asymptotic ansatz is derived using saddle point techniques. This corresponds to the asymptotic ansatz discussed in [1, 2]. Section 4 ends with conclusions and open questions.

## 2 The Model, Notation and Conventions

We consider models of the type (see [14] for details of notation and definitions)

$$Z = \exp(-F) = \int dM e^{-N \text{Tr} V(M)} \quad (2.1)$$

where  $V(M) = V_0(M) - \frac{t}{2} \ln M^2$  and here  $V_0(M) = \frac{1}{2}\mu M^2$ . See Fig. 1.

In general the orthogonal polynomials are  $P_n(\lambda) = \lambda^n + l.o.$  where  $\lambda$  are the eigenvalues of  $M$  and the orthogonality conditions are  $\int_{-\infty}^{\infty} d\lambda e^{-NV(\lambda)} P_n(\lambda) P_m(\lambda) = h_n \delta_{nm}$ . The partition function can be expressed in terms of the  $h_n$ 's as  $Z = N! h_0 h_1 h_2 \dots h_{N-1}$ .

For the large  $N$  limit, the density of eigenvalues,  $\rho(z) \equiv (\frac{1}{N}) \sum_{i=1}^N \delta(z - \lambda_i)$ , can be found by solving either a saddle point equation or the Schwinger-Dyson equation. In terms of the generating function  $F(z) = \frac{1}{N} \left[ \text{tr} \frac{1}{z-M} \right] \rightarrow$

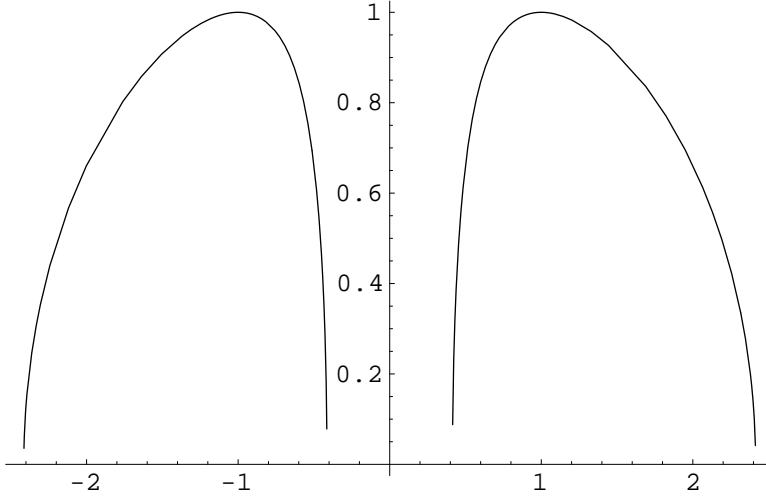


Figure 2: The density of eigenvalues for the Gaussian Penner Random Matrix Model for  $\mu = 1$  and  $t = 1$ .

$\int dz' \frac{\rho(z')}{z-z'}$  the Schwinger-Dyson equation reads  $F(z)^2 - V'(z)F(z) = M(z)$ , where  $M(z)$  is a meromorphic function. The density of eigenvalues is given by  $\rho(z) = -\frac{1}{\pi} \text{Im} F(z)$ . The generating function  $F(z)$  in the large  $N$  limit for  $t > 0$  is

$$F(z) = \frac{\mu z}{2} - \frac{t}{2z} - \frac{\mu}{2z} \sqrt{(z^2 - a^2)(z^2 - b^2)} \quad (2.2)$$

where  $a^2 = \frac{(2+t)}{\mu} + \frac{2}{\mu} \sqrt{(1+t)}$  and  $b^2 = \frac{(2+t)}{\mu} - \frac{2}{\mu} \sqrt{(1+t)}$ . See Fig. 2 for the corresponding density of eigenvalues.

### 3 Exact Solution for the Symmetric Gaussian Model

The orthogonal polynomials satisfy the recurrence relation

$$(z - S_n)P_n(z) = P_{n+1}(z) + R_n P_{n-1}(z) \quad (3.1)$$

where  $S_n, R_n$  are the recurrence coefficients. For symmetric models  $S_n = 0$ . For even potentials instead of recurrence relation

$$zP_n(z) = P_{n+1}(z) + R_n P_{n-1}(z) \quad (3.2)$$

one can use

$$z^2 P_{2n}(z) = P_{2n+2}(z) + (R_{2n+1} + R_{2n})P_{2n}(z) + R_{2n-1}R_{2n}P_{2n-2}(z) \quad (3.3)$$

and

$$z^2 P_{2n+1}(z) = P_{2n+3}(z) + (R_{2n+1} + R_{2n+2})P_{2n+1}(z) + R_{2n+1}R_{2n}P_{2n-1}(z). \quad (3.4)$$

Where we have multiplied  $z$  to Eq. (3.2) and expanded. Then eq. (3.3) contains only even polynomials  $P_{2n}(-z) = P_{2n}(z)$  and eq. (3.4) contains only odd polynomials  $P_{2n+1}(-z) = -P_{2n+1}(z)$ . This simplifies the solution as we will see.

(1). Let us first work with the even set. Let  $y = z^2$  and define functions  $\mathcal{P}_n(y) = P_{2n}(z)$ . In terms of these ‘new’ polynomials

$$y\mathcal{P}_n(y) = \mathcal{P}_{n+1}(y) + \mathcal{S}_n\mathcal{P}_n(y) + \mathcal{R}_n\mathcal{P}_{n-1}(y) \quad (3.5)$$

where  $\mathcal{S}_n = R_{2n} + R_{2n+1} = A_n + B_n$  and  $\mathcal{R}_n = R_{2n-1}R_{2n} = A_nB_{n-1}$ . These polynomials obey

$$\int_0^\infty dy e^{-N'[\mathcal{V}_0(y) - t' \log y]} \mathcal{P}_n(y) \mathcal{P}_m(y) = h_n \delta_{n,m} \quad (3.6)$$

where  $\mathcal{V}_0(y) = 2V_0(z) = \mu y + \dots$ ,  $t' = t - \frac{1}{2N'}$  and  $N' = \frac{N}{2}$ .

This is the same as the brick wall problem i.e. the linear Penner model if  $t \leftrightarrow t'$ ,  $N \leftrightarrow N'$  and  $t'N' = (t - \frac{1}{N})\frac{N}{2} = \frac{(Nt-1)}{2}$ .

(2). A similar analysis can be carried out for the odd set. Define

$$\bar{\mathcal{P}}_n(y) = z^{-1}P_{2n+1}(z). \quad (3.7)$$

Then

$$y\bar{\mathcal{P}}_n(y) = \bar{\mathcal{P}}_{n+1}(y) + \bar{\mathcal{S}}_n\bar{\mathcal{P}}_n(y) + \bar{\mathcal{R}}_n\bar{\mathcal{P}}_{n-1}(y) \quad (3.8)$$

where

$$\bar{\mathcal{S}}_n = R_{2n+1} + R_{2n+2} \quad (3.9)$$

and

$$\bar{\mathcal{R}}_n = R_{2n+1}R_{2n}. \quad (3.10)$$

Because of the extra factor of  $z$  associated with the odd series the ‘barred’ polynomials satisfy orthogonality condition

$$\int_0^\infty dy e^{-N'[\mathcal{V}_0(y) - \bar{t}' \log y]} \bar{\mathcal{P}}_n(y) \bar{\mathcal{P}}_m(y) = \bar{h}_n \delta_{n,m} \quad (3.11)$$

where  $\bar{t}' = t + \frac{1}{2N'}$ . This barred system can be solved as a brick-wall problem. Original recurrence coefficients are obtained by

$$R_{2n+1} = \frac{1}{2} \{ \mathcal{S}_n + \sqrt{\mathcal{S}_n^2 - 4\bar{\mathcal{R}}_n} \} \quad (3.12)$$

$$R_{2n} = \frac{1}{2} \{ \mathcal{S}_n - \sqrt{\mathcal{S}_n^2 - 4\bar{\mathcal{R}}_n} \}. \quad (3.13)$$

Using  $\mathcal{W}_{n+1} + \mathcal{W}_n + \mathcal{S}_n \mathcal{Y}_n = \frac{2n+1+Nt}{N}$  and  $\mathcal{S}_n \mathcal{W}_{n+1} - \mathcal{W}_n - \frac{1}{N} = -R_{n+1} \mathcal{Y}_{n+1} + R_n \mathcal{Y}_{n-1}$  (see ref. [14] for details and definitions of  $\mathcal{W}$  and  $\mathcal{Y}$ ) we get

$$\mathcal{S}_n = \frac{2n+1+t'N'}{\mu N'} = \frac{4n+1+tN}{\mu N} \quad (3.14)$$

$$\bar{\mathcal{S}}_n = \frac{2n+1+\bar{t}'N'}{\mu N'} = \frac{4n+3+tN}{\mu N} \quad (3.15)$$

$$\mathcal{R}_n = \frac{n(n+t'N')}{\mu^2 N'^2} = \frac{2n(2n-1+tN)}{\mu^2 N^2} \quad (3.16)$$

$$\bar{\mathcal{R}}_n = \frac{n(n+\bar{t}'N')}{\mu^2 N'^2} = \frac{2n(2n+1+tN)}{\mu^2 N^2}. \quad (3.17)$$

(1). For the even set the orthogonality relation

$$\int_0^\infty dy e^{-N'[\mathcal{V}_0(y) - t' \log y]} \mathcal{P}_n(y) \mathcal{P}_m(y) = h_n \delta_{n,m} \quad (3.18)$$

simplifies to

$$\int_0^\infty dy e^{-N' \mathcal{V}_0(y)} y^{N't'} \mathcal{P}_n(y) \mathcal{P}_m(y) = h_n \delta_{n,m}. \quad (3.19)$$

Now as

$$\mathcal{R}_n = \frac{n(n+t'N')}{\mu^2 N'^2} = \frac{2n(2n-1+tN)}{\mu^2 N^2}, \quad (3.20)$$

$$\mathcal{R}_1 = \frac{2(2-1+tN)}{\mu^2 N^2} = \frac{2(1+tN)}{\mu^2 N^2}, \quad (3.21)$$

$$\mathcal{R}_2 = \frac{4(4-1+tN)}{\mu^2 N^2} \quad (3.22)$$

etc. Note the following

$$\mathcal{R}_n = \frac{h_n}{h_{n-1}}. \quad (3.23)$$

Thus  $h_n$  is

$$\begin{aligned} h_n &= \mathcal{R}_n h_{n-1} = \mathcal{R}_n \mathcal{R}_{n-1} h_{n-2} \\ &= \mathcal{R}_n \mathcal{R}_{n-1} \mathcal{R}_{n-2} \dots \mathcal{R}_1 h_0 \\ &= \frac{n(n+t'N')}{\mu^2 N'^2} \frac{(n-1)(n-1+t'N')}{\mu^2 N'^2} \\ &\quad \dots \frac{(1+t'N')}{\mu^2 N'^2} h_0 \\ &= \frac{n!}{(\mu N')^{2n}} \frac{\Gamma(n+t'N'+1)}{\Gamma(t'N'+1)} h_0. \end{aligned} \quad (3.24)$$

Note:

$$\mathcal{P}_n(y) = P_{2n}(z) \quad (3.25)$$

for  $n = 0$

$$\mathcal{P}_n(y) = P_0(z) = 1. \quad (3.26)$$

Then

$$\begin{aligned} h_0 &= \int_0^\infty dy e^{-N'\mathcal{V}_0(y)} y^{N't'} \mathcal{P}_0(y) \mathcal{P}_0(y) \\ &= \int_0^\infty dy e^{-N'\mathcal{V}_0(y)} y^{N't'}. \end{aligned} \quad (3.27)$$

For

$$\mathcal{V}_0(y) = \mu y, \quad (3.28)$$



$$h_0 = \int_0^\infty dy e^{-N'\mu y} y^{N't'}. \quad (3.29)$$

Defining

$$u = N'\mu y, \quad (3.30)$$

$$du = N'\mu dy \quad (3.31)$$

we get

$$\begin{aligned} h_0 &= \int_0^\infty \frac{du}{N'\mu} e^{-u} \left(\frac{u}{N'\mu}\right)^{N't'} \\ &= \frac{1}{(N'\mu)^{N't'+1}} \Gamma(N't' + 1) \end{aligned} \quad (3.32)$$

and

$$h_n = \frac{n! \Gamma(n + 1 + t'N')}{(\mu N')^{2n + N't' + 1}}. \quad (3.33)$$

$$(3.34)$$

Rewritting

$$\mathcal{P}_n(y) = Q_n(u) \quad (3.35)$$

the orthogonality condition is

$$\begin{aligned} \int_0^\infty du u^{N't'} e^{-u} Q_n(u) Q_m(u) &= (N'\mu)^{N't'+1} h_n \delta_{n,m} \\ &= \frac{(n!)^2 \Gamma(n + 1 + t'N')}{(\mu N')^{2n} \Gamma(n + 1)} \delta_{n,m}. \end{aligned}$$

Redefine  $Q'_n(u) = \frac{(\mu N')^n}{n!} Q_n(u)$ . Then

$$\int_0^\infty du u^{N't'} e^{-u} Q'_n(u) Q'_n(u) = \frac{\Gamma(n + 1 + t'N')}{\Gamma(n + 1)} \delta_{n,m}, \quad (3.36)$$

comparing with the Associated Laguerre Polynomials  $L_n^{(\alpha)}(x)$  we get

$$Q'_n(u) = (-1)^n L_n^{N't'}(u), \quad (3.37)$$

then ( $\alpha = N't'$  Associated Laguerre Polynomials)

$$\mathcal{P}_n(y) = Q_n(u) = \frac{n!}{(\mu N')^n} (-1)^n L_n^{N't'}(u) \quad (3.38)$$

and

$$\begin{aligned} \hat{\mathcal{P}}_n(y) &= \frac{\mathcal{P}_n(y)}{\sqrt{h_n}} \\ &= \sqrt{\frac{(\mu N')^{N't'+1}}{(n!) \Gamma(n+1+t'N')}} (n!) (-1)^n L_n^{N't'}(N'\mu y). \end{aligned} \quad (3.39)$$

Thus the normalized even set of orthogonal polynomials are

$$\begin{aligned} \psi_{2n}(y) &= e^{-\frac{N'}{2}[\mu y - t' \log y]} \hat{\mathcal{P}}_n(y) \\ &= \left[ \frac{n! (N'\mu)^{N't'+1}}{\Gamma(n+1+N't')} \right]^{\frac{1}{2}} y^{\frac{N't'}{2}} e^{-\frac{N'\mu y}{2}} L_n^{N't'}(N'\mu y). \end{aligned} \quad (3.40)$$

(2). For odd set

$$\int_0^\infty dy e^{-N'[\mu y - \bar{t}' \log y]} \bar{\mathcal{P}}_n(y) \bar{\mathcal{P}}_m(y) = \bar{h}_n \delta_{nm} \quad (3.41)$$

where  $\bar{t}' = t + \frac{1}{2N'}$  and  $N'\bar{t}' = \frac{Nt+1}{2}$ . The orthogonality condition are

$$\int_0^\infty dy e^{-N'\mu y} y^{N'\bar{t}'} \bar{\mathcal{P}}_n(y) \bar{\mathcal{P}}_m(y) = \bar{h}_n \delta_{nm}. \quad (3.42)$$

Note that

$$\bar{\mathcal{R}}_n = \frac{\bar{h}_n}{\bar{h}_{n-1}}. \quad (3.43)$$

For

$$\bar{\mathcal{R}}_n = \frac{n(n+\bar{t}')}{\mu^2 N'^2}. \quad (3.44)$$

we get

$$\begin{aligned}
\bar{h}_n &= \bar{\mathcal{R}}_n \bar{h}_{n-1} \\
&= \bar{\mathcal{R}}_n \bar{\mathcal{R}}_{n-1} \bar{\mathcal{R}}_{n-2} \dots \bar{\mathcal{R}}_1 \bar{h}_0 \\
&= \frac{n!(n + \bar{t}' N')!}{(\mu N')^{2n} (\bar{t}' N')!} \bar{h}_0.
\end{aligned} \tag{3.45}$$

Note:

$\bar{\mathcal{P}}_n(y) = z^{-1} P_{2n+1}(z)$  and  $\bar{\mathcal{P}}_0(y) = z^{-1} P_1(z)$ .

But  $P_1(z) = z$  hence  $\bar{\mathcal{P}}_0(y) = z^{-1} z = 1$ . Therefore

$$\begin{aligned}
\bar{h}_0 &= \int_0^\infty dy e^{-N' \mu y} y^{N' \bar{t}'} \bar{\mathcal{P}}_0(y) \bar{\mathcal{P}}_0(y) \\
&= \int_0^\infty dy e^{-N' \mu y} y^{N' \bar{t}'}.
\end{aligned} \tag{3.46}$$

Defining

$$u = N' \mu y \tag{3.47}$$

and

$$du = N' \mu dy, \tag{3.48}$$

we get

$$\begin{aligned}
\bar{h}_0 &= \int_0^\infty \frac{du}{N' \mu} e^{-u} \left( \frac{u}{N' \mu} \right)^{N' \bar{t}'} \\
&= \frac{1}{(N' \mu)^{N' \bar{t}'+1}} \int_0^\infty du e^{-u} u^{N' \bar{t}'}.
\end{aligned} \tag{3.49}$$

So that

$$\bar{h}_n = \frac{\Gamma(n+1) \Gamma(n+1 + \bar{t}' N')}{(N' \mu)^{2n + N' \bar{t}'+1}}. \tag{3.50}$$

Now let  $\bar{Q}_n(u) = \bar{\mathcal{P}}_n(y)$ . Therefore

$$\begin{aligned}
\int_0^\infty du e^{-u} u^{N'\bar{t}'} \bar{Q}_n(u) \bar{Q}_m(u) &= (N'\mu)^{N'\bar{t}'+1} \bar{h}_n \delta_{nm} \\
&= \frac{\Gamma(n+1)\Gamma(n+1+\bar{t}'N')}{(\mu N')^{2n}} \delta_{nm} \\
&= \left(\frac{n!}{(\mu N')^n}\right)^2 \frac{\Gamma(n+1+\bar{t}'N')}{n!} \delta_{nm}. \quad (3.51)
\end{aligned}$$

Define  $\bar{Q}'_n(u) = \frac{(\mu N')^n}{n!} \bar{Q}_n(u)$  then

$$\int_0^\infty du e^{-u} u^{N'\bar{t}'} \bar{Q}'_n(u) \bar{Q}'_m(u) = \frac{\Gamma(n+1+\bar{t}'N')}{n!} \delta_{nm}. \quad (3.52)$$

For

$$\begin{aligned}
\bar{Q}'_n(u) &= \frac{(\mu N')^n}{n!} Q_n(u) \\
&= (-1)^n L_n^{N'\bar{t}'}(u)
\end{aligned} \quad (3.53)$$

where  $\alpha = N'\bar{t}'$ . Now

$$\bar{\mathcal{P}}_n(y) = \bar{Q}_n(u) = \frac{n!(-1)^n}{(\mu N')^n} L_n^{N'\bar{t}'}(N'\mu y). \quad (3.54)$$

hence

$$\begin{aligned}
\hat{\bar{\mathcal{P}}}_n(y) &= \frac{\bar{\mathcal{P}}_n(y)}{\sqrt{\bar{h}_n}} \\
&= \sqrt{\frac{(\mu N')^{N'\bar{t}'+1} n!}{\Gamma(n+1+\bar{t}'N')}} (-1)^n L_n^{N'\bar{t}'}(N'\mu y). \quad (3.55)
\end{aligned}$$

Thus the normalized odd orthogonal polynomials are

$$\begin{aligned}
\psi_{2n+1}(y) &= e^{-\frac{N'}{2}[\mu y - \bar{t}' \log y]} \hat{\bar{\mathcal{P}}}_n(y) \\
&= \left[\frac{n!(N'\mu)^{N'\bar{t}'+1}}{\Gamma(n+1+N'\bar{t}')}\right]^{\frac{1}{2}} y^{\frac{N'\bar{t}'}{2}} e^{-\frac{N'\mu y}{2}} L_n^{N'\bar{t}'}(N'\mu y). \quad (3.56)
\end{aligned}$$

The important lesson to learn from this exercise is that  $\alpha = \frac{(Nt - (-1)^{n'})}{2}$  for  $n'$  an integer which is even or odd and  $x = N'\mu y$  for the orthogonal polynomials, which are proportional to the generalized Laguerre polynomials  $L_n^{(\alpha)}(x)$ , of the Gaussian Penner Matrix Model.

## 4 The Old Asymptotic Formula for Associative Laguerre Ensemble

Consider the asymptotic formula given in Szego's book on orthogonal polynomials. The formulas of Plancherel-Rotach type for Laguerre polynomials for  $\alpha$  arbitrary and real,  $\epsilon$  fixed positive number for  $x = (4n + 2\alpha + 2) \cos^2 \phi$ ,  $\epsilon \leq \phi \leq \frac{\pi}{2} - \epsilon n^{\frac{1}{2}}$  are given below

$$e^{\frac{-x}{2}} L_n^{(\alpha)}(x) = (-1)^n (\pi \sin \phi)^{\frac{-1}{2}} x^{-\frac{\alpha}{2} - \frac{1}{4}} n^{\frac{\alpha}{2} - \frac{1}{4}} \{ \cos[(n + \frac{(\alpha + 1)}{2})(\sin 2\phi - 2\phi) + \frac{\pi}{4}] + (nx)^{\frac{-1}{2}} O(1) \}. \quad (4.1)$$

For the Gaussian Penner model with  $\alpha = \frac{(Nt - (-1)^{n'})}{2}$  and  $x = N'\mu y$

$$e^{\frac{-x}{2}} L_n^{(\alpha)}(x) = (-1)^n (\pi \sin \phi)^{\frac{-1}{2}} x^{-\frac{\alpha}{2} - \frac{1}{4}} n^{\frac{\alpha}{2} - \frac{1}{4}} \{ \cos[(n + \frac{1}{2})(\sin 2\phi - 2\phi) + \frac{(Nt - (-1)^{n'})}{4}(\sin 2\phi - 2\phi) + \frac{\pi}{4}] + (nx)^{\frac{-1}{2}} O(1) \}. \quad (4.2)$$

The combined asymptotic ansatz for the even and odd results eq. (3.40) and eq. (3.56) for the Gaussian Penner model in this asymptotic region is ( $n'$  stands for both even  $2n$  and odd  $2n + 1$  integers)

$$\psi_{n'}(x) = e^{\frac{-x}{2}} L_n^{(\alpha)}(x) = (-1)^n (\pi \sin \phi)^{\frac{-1}{2}} x^{-\frac{\alpha}{2} - \frac{1}{4}} n^{\frac{\alpha}{2} - \frac{1}{4}} \{ \cos[(n + \frac{1}{2} + \frac{Nt}{4})(\sin 2\phi - 2\phi) + \frac{\pi}{4} - \frac{(-1)^{n'}}{4}(\sin 2\phi - 2\phi)] + (nx)^{-\frac{1}{2}} O(1) \} \quad (4.3)$$

Note that the term  $\frac{(-1)^{n'}}{4}(\sin 2\phi - 2\phi)$  is not the extra term (with  $\eta$  see ref. [1, 2] or eq. (5.15) below) that was found for the asymptotic ansatz in the double well problem.

## 5 The New Asymptotic Formula for Associative Laguerre Ensemble

Let us start with an expression which has a well defined  $\alpha = 0$  limit. Using

$$\frac{d}{dx}f(x) = \frac{d}{du}f(u+x)|_{u=0} \quad (5.1)$$

we can write using Cauchy's theorem

$$\begin{aligned} L_n^{(\alpha)}(x) &= \frac{e^x}{n!} x^{-\alpha} \left( \frac{d}{du} \right)^n [(x+u)^{n+\alpha} e^{-(x+u)}]_{u=0} \\ &= \frac{x^{-\alpha}}{n!} \left( \frac{d}{du} \right)^n [(x+u)^{(n+\alpha)} e^{-(u)}]_{u=0} \\ &= x^{-\alpha} \int_c \frac{dz}{2i\pi} \frac{1}{z^{n+1}} (z+x)^{n+\alpha} e^{-z} \end{aligned} \quad (5.2)$$

in which the countour C is a small circle around the origin ( $|z| < x$ ) change  $z \rightarrow nz$  ( $|z| < \frac{x}{n}$ )

$$\begin{aligned} L_n^{(\alpha)}(x) &= \left( \frac{x}{n} \right)^{-\alpha} \int_c \frac{dz}{2i\pi} \frac{1}{z^{n+1}} \left( z + \frac{x}{n} \right)^{n+\alpha} e^{-nz} \\ &= \left( \frac{x}{n} \right)^{-\alpha} \int_c \frac{dz}{2i\pi} \frac{1}{z} e^{-nf(z)} \end{aligned} \quad (5.3)$$

$$f(z) = z + \log z - \left( 1 + \frac{\alpha}{n} \right) \log \left( z + \frac{x}{n} \right) \quad (5.4)$$

we explore  $\frac{\alpha}{n}$  and  $\frac{x}{n}$  finite. In this representation the limit  $\alpha = 0$  is well defined. C is a circle around the origin. The saddle point is given by an expression

$$z^2 + z \left( \frac{x-\alpha}{n} \right) + \frac{x}{n} = 0. \quad (5.5)$$

With the parametrization

$$\frac{\alpha-x}{n} = 2\sqrt{\frac{x}{n}} \cos \phi \quad (5.6)$$

$$z_0 = \sqrt{\frac{x}{n}} e^{i\phi} \quad (5.7)$$

and  $\bar{z}_0$  are the saddle points. This is valid in the range  $|\frac{\alpha-x}{n}| < 2\sqrt{\frac{x}{n}}$ ; if the parameters are such that this inequality is reversed, there is only one real saddle-point. This is where the new saddle points will have to be taken into account.

If there is a saddle-point in the integral

$$I = \int_c \frac{dz}{2i\pi} \frac{1}{z} e^{-nf(z)} \quad (5.8)$$

at  $z_0$  then

$$I \approx \frac{e^{-nf(z_0)}}{z_0} \sqrt{\frac{2\pi}{n|f''(z_0)|}} e^{\frac{-i\theta}{2}} \quad (5.9)$$

with  $f''(z_0) = |f''(z_0)|e^{i\theta}$ . Indeed expand  $f(z)$  around  $z_0$

$$f(z) = f(z_0) + \frac{1}{2}(z - z_0)^2 |f''(z_0)| e^{i\theta} \quad (5.10)$$

path is  $z - z_0 = xe^{\frac{-i\theta}{2}}$

$$I \approx e^{-i\frac{\theta}{2} - nf(z_0)} \int_{-\infty}^{+\infty} dx e^{-\frac{n}{2}x^2 |f''(z_0)|}. \quad (5.11)$$

Here we need to add the contributions of  $z_0$ ,  $\bar{z}_0$  and the other saddle points. Therefore if  $f(z_0) = A + iB$

$$I \approx e^{-nA} \sqrt{\frac{2\pi}{x|f''(z_0)|}} 2 \cos(nB + \phi + \frac{\theta}{2})(x) \quad (5.12)$$

with

$$f''(z_0) = \sqrt{\frac{n}{x}} \frac{2i \sin\phi}{\sqrt{\frac{x}{n}} + e^{i\phi}} e^{-i\phi} \quad (5.13)$$

and  $\theta$  is it's phase. The  $(-1)^n$  would only show up in B. This is a new asymptotic expansion of the generalized Laguerre ensemble in a novel asymptotic regime. It would be nice to have a physical picture of this asymptotic regime.

On simplifying the above expression it will give the asymptotic ansatz found in ref.[1] for the double well matrix model. Following ref. [1] for  $N$  large but  $N - n \approx O(1)$  and  $x$  lying in the two cuts the asymptotic ansatz for the orthogonal polynomials of the Gaussian Penner Matrix Model can be approximated by

$$\psi_n(x) = \frac{1}{\sqrt{f}} \left[ \cos(N\zeta - (N - n)\phi + \chi + (-1)^n \eta)(x) + O\left(\frac{1}{N}\right) \right] \quad (5.14)$$

where  $f, \zeta, \phi, \chi$  and  $\eta$  are functions of  $x$  and are given by

$$\begin{aligned} f(x) &= \frac{\pi}{2x} \frac{(b^2 - a^2)}{2} \sin 2\phi(x) \\ \zeta'(x) &= -\pi\rho(x) \\ \cos 2\phi(x) &= \frac{x^2 - \frac{(a^2 + b^2)}{2}}{\frac{(b^2 - a^2)}{2}} \\ \cos 2\eta(x) &= b \frac{\cos \phi(x)}{x} \\ \sin 2\eta(x) &= a \frac{\sin \phi(x)}{x} \\ \chi(x) &= \frac{1}{2}\phi(x) - \frac{\pi}{4} \end{aligned} \quad (5.15)$$

with  $a^2$  and  $b^2$  as given in sec. (2) for the Gaussian Penner model.

All the corresponding correlation functions of this model will be as obtained in ref. [1] and ref. [2]. Following ref. [2] and using the contour of integration for the Gaussian Penner model, see Fig. 3, the smoothed density-density correlation function can be derived in the thermodynamic limit and is an oscillating function of  $N$ :

$$\begin{aligned} 2\pi^2 N^2 \rho_2^\epsilon(\lambda, \mu) &= \frac{\epsilon_\lambda \epsilon_\mu}{\beta \sqrt{|\sigma(\lambda)|} \sqrt{|\sigma(\mu)|}} \frac{1}{(\mu - \lambda)^2} \\ &\quad \left( \lambda\mu(\lambda\mu - a^2 - b^2) + a^2 b^2 + (-1)^N ab(\mu - \lambda)^2 \right). \end{aligned} \quad (5.16)$$

Here for the symmetric Gaussian Penner Model  $\sigma(z) = (z^2 - a^2)(z^2 - b^2)$ ,  $a^2 = \frac{(2+t)}{\mu} + \frac{2}{\mu}\sqrt{(1+t)}$ ,  $b^2 = \frac{(2+t)}{\mu} - \frac{2}{\mu}\sqrt{(1+t)}$ ,  $\epsilon_\lambda = +1$  for  $b < \lambda < a$ ,



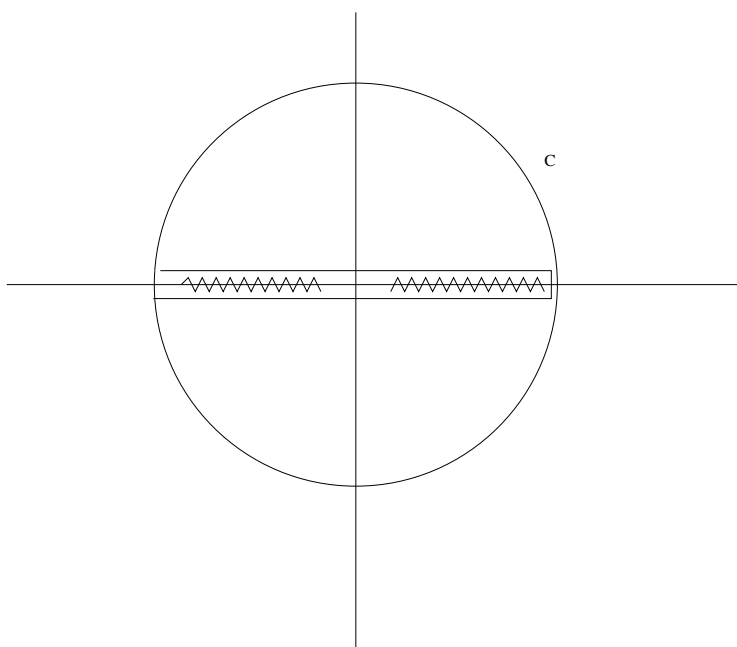


Figure 3: The contour of integration for the two-cut Gaussian Penner model.

$\epsilon_\lambda = -1$  for  $-a < \lambda < -b$  and  $\beta = 1, 2, 4$  depending on whether  $M$  the matrix is real orthogonal, hermitian or self-dual quaternionian.

Using this expression in the formulas for the mesoscopic fluctuation [15] would give rise to terms depending on  $N$ . This may be observed in single electron experiments on mesoscopic samples which have gaps in their eigenvalue spectrum. Work to explicitly obtain all these expressions in the Chiral Matrix Model in this asymptotic regime is in progress.

## 6 Conclusion

Finally let us note that in matrix models when the number of connected components for the support of the eigenvalues changes, one finds a new universality class for the correlators which has been extended here to include the non-polynomial potentials. It is not completely obvious that it is legitimate to use the simple one cut function in the application to mesoscopic fluctuations as the correlation function are different for these gapped random matrix models because of single electron tunneling effects. Further questions of symmetry breaking in these singular random matrix models are open questions to be explored in the future.

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